Math 565: Functional Analysis
Lecture 12

Existence of Banach limits. $\exists\ \tilde{L} \in (\ell^\infty)^*$ such that

(i) $\tilde{L} \circ S = \tilde{L}$ (shift-invariant);
(ii) $\|\tilde{L}\| = 1$;
(iii) $\tilde{L}|_c = L$;
(iv) $\tilde{L}$ is positive, i.e. $x \geq 0 \Rightarrow \tilde{L}x \geq 0$. In particular, if $x \in \ell^\infty_{\mathbb{R}}$, then $Lx \in \mathbb{R}$.

Proof. The "in particular" part in (iv) follows from the nonnegativity of $\tilde{L}$ because if $x \in \ell^\infty_{\mathbb{R}}$, then $x = x_+ - x_-$ with $x_+, x_- \geq 0$, hence $\tilde{L}(x) = \tilde{L}(x_+) - \tilde{L}(x_-) = \text{nonneg} - \text{nonneg} \in \mathbb{R}$.

We first define $\tilde{L}$ on $\ell^\infty_{\mathbb{R}}$, and then uniquely extend (by linearity) to $\ell^\infty_{\mathbb{C}}$.
Let $Y := \{x - S(x) : x \in \ell^\infty_{\mathbb{R}}\}$, the set of "coboundaries", which is easily a subspace.

Claim. $\operatorname{dist}(\mathbb{1}, Y) =: \|\mathbb{1} + Y\| = 1 = \operatorname{dist}(\mathbb{1}, \bar{Y})$.

Pf of Claim. $\|\mathbb{1} + Y\| \leq \|\mathbb{1}\|_\infty = 1$. For the converse, fix $y = x - S(x) \in Y$. Because $\|\mathbb{1} - y\|_\infty \geq |1 - y(n)|$ for all $n \in \mathbb{N}$, if $\exists\, n_0 \in \mathbb{N}$ such that $y(n_0) \leq 0$, then $\|\mathbb{1} - y\|_\infty \geq |1 - y(n_0)| = 1 + |y(n_0)| \geq 1$.
Suppose $\forall n \in \mathbb{N}$, $y(n) > 0$. But $y = x - S(x)$, so $y(n) = x(n) - x(n+1) > 0$ means $x(n) > x(n+1)$.
Since $x \in \ell^\infty_{\mathbb{R}}$, i.e. bounded, monotone convergence thm implies $\lim_{n\to\infty} y_n = \lim_{n\to\infty}(x_n - x_{n+1}) \to 0$.
Thus, $\|\mathbb{1} - y\|_\infty \geq |1 - y_n| \to 1$ as $n \to \infty$, so $\|\mathbb{1} - y\|_\infty \geq 1$. □

By Hahn-Banach for annihilating subspaces, $\exists\ \hat{L} \in (\ell^\infty_{\mathbb{R}})^*$ with $\hat{L}|_{\bar{Y}} = 0$, $\|\hat{L}\| = 1$, and $\hat{L}(\mathbb{1}) = \|\mathbb{1} + Y\| = 1$.

(iii) Since $c_{\mathbb{R}} = c_{0,\mathbb{R}} + \mathbb{R} \cdot \mathbb{1}$ and $\hat{L}(\mathbb{1}) = 1$, it is enough to show that $\hat{L}|_{c_{0,\mathbb{R}}} = 0$.
If $x \in c_{0,\mathbb{R}}$, then $\forall \varepsilon > 0$ and large enough $n \in \mathbb{N}$, $\|S^n(x)\| < \varepsilon$, hence $|\hat{L}x| = |\hat{L}(S^n x)| \leq$

$\|L\| \cdot \|s^n(x)\| < \varepsilon$, so $\hat{L}x = 0$. Another way to see this is that $c_{c,\mathbb{R}} \subseteq c_{0,\mathbb{R}}$ is dense and clearly $\hat{L}|_{c_{c,\mathbb{R}}} = 0$ because for each $x \in c_{c,\mathbb{R}}$ $\exists n \in \mathbb{N}$ s.t. $s^n(x) = 0$, hence $\hat{L}x = \hat{L}s^n(x) = 0$.

(iv) Suppose $x \in \ell^\infty_{\mathbb{R}}$ is nonnegative but $\hat{L}x < 0$. Replacing x with $\frac{1}{\|x\|}x$, we may assume $\|x\|_\infty = 1$. Then $\forall n$ $0 \le x(n) \le 1$, hence $\|\mathbb{1} - x\|_\infty = \sup_n |1 - x(n)| \le 1$ so we must have $|\hat{L}(\mathbb{1} - x)| = |1 - \hat{L}(x)| > 1 = \|L\| \cdot \|x\|_\infty$, a contradiction.

Now define $\tilde{L}$ on $\ell^\infty_{\mathbb{C}} = \ell^\infty_{\mathbb{R}} + i\,\ell^\infty_{\mathbb{R}}$ by $\tilde{L}x := \hat{L}(\mathrm{Re}\,x) + i\hat{L}(\mathrm{Im}\,x)$. All of (i), (iii), (iv) still hold by linearity, but instead of (ii) we only have $\|\tilde{L}\| \le \|\hat{L}\| + \|\hat{L}\| = 2$. We show that $\|\tilde{L}\| = 1$ by proving $\|\tilde{L}\| \le 1$. Since $\tilde{L}$ (has norm ≤ 2) is continuous, it suffices to show $\|\tilde{L}\| \le 1$ on a dense subset of $\ell^\infty_{\mathbb{C}}$. Let

$$S := \{x \in \ell^\infty_{\mathbb{C}} : x(\mathbb{N}) \text{ is finite, i.e. } x \text{ is a simple function}\}.$$

Recall that for a bdd $f \in \ell^\infty_{\mathbb{C}}$ $\exists$ simple functions $(f_n) \subseteq \ell^\infty_{\mathbb{C}}$ with $f_n \to f$ uniformly and $|f_n| \nearrow |f|$, hence $f_n \to f$ in ℓ^∞ norm. Thus, S is dense in ℓ^∞.

Let $x \in S$, so $x = \sum_{i<k} \alpha_i \mathbb{1}_{A_i}$, where $\alpha_i \in \mathbb{C}$ and $\mathbb{N} = \bigsqcup_{i<k} A_i$. But

$$1 = \tilde{L}(\mathbb{1}) = \tilde{L}(\mathbb{1}_{\mathbb{N}}) = \sum_{i<k} \tilde{L}(\mathbb{1}_{A_i})$$

hence $A \mapsto \tilde{L}(\mathbb{1}_A)$ is a finitely additive measure on $\mathcal{P}(\mathbb{N})$. Thus, $\tilde{L}x = \sum_{i<k} \alpha_i \tilde{L}(\mathbb{1}_{A_i})$ = weighted average of the α_i because $\tilde{L}(\mathbb{1}_{A_i}) \ge 0$ and add up to 1.

Hence $|\tilde{L}x| \le \max_{i<k} |\alpha_i| = \|x\|_\infty$. Hence $\|\tilde{L}\| < 1$ on S, thus on $\ell^\infty_{\mathbb{C}}$. QED

Remark. The last theorem gives another proof that ℓ^1 is not reflexive by exhibiting a concrete element $\tilde{L}$ of $(\ell^\infty)^* \setminus \hat{\ell^1}$. Indeed, for all $f \in \hat{\ell^1}$, $f \circ s \neq f$ because $\lim_{n\to\infty} f(n) = 0$.

In other words, $\tilde{L}$ defines a finitely additive measure on $\mathcal{P}(\mathbb{N})$ which is shift-invariant (thus, proving that $\mathbb{N}$ is an amenable semigroup).

Bounded linear functionals $\tilde{L}$ as in the theorem are called Banach limits or means on ℓ^∞.

There are other linear functionals in $(\ell^\infty)^*$, i.e. other finitely additive measures on $\mathcal{P}(\mathbb{N})$, which are not shift invariant, but using them, one could construct means.

Def. A finitely additive 0/1-valued measure on $\mathcal{P}(\mathbb{N})$ is called an ultrafilter-measure.

Trivial examples. Dirac measures δ_n, $n \in \mathbb{N}$, are ultrafilter-measures.

The name ultrafilter comes from looking at the collection $\mathcal{F}_\mu$ of measure 1 sets of an ultrafilter-measure μ on $\mathcal{P}(\mathbb{N})$. This set $\mathcal{F}_\mu$ satisfies:
(i) $\mathcal{F}_\mu$ is upward closed: $A \in \mathcal{F}$ and $A \subseteq B \Rightarrow B \in \mathcal{F}$.
(ii) $\mathcal{F}_\mu$ is closed under finite intersections (hence finite unions of μ-null sets are μ-null).
(iii) For every $A \subseteq \mathbb{N}$, exactly one of A or A^c is in $\mathcal{F}_\mu$.
Collections of sets satisfying (i) and (ii) are called filters (the dual of ideals)
Property (iii) makes the filter ultra. Call a filter principal if it contains a singleton.
Thus, $\mathcal{F}_\mu$ is principal exactly when μ is a Dirac measure.

Conversely, given any filter $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$, note that $\mathcal{A} := \mathcal{F} \cup \hat{\mathcal{F}}$, where $\hat{\mathcal{F}} := \{A \subseteq \mathbb{N} : A^c \in \mathcal{F}\}$, is an algebra, and $\mu_{\mathcal{F}} : \mathcal{A} \to \{0,1\}$ defined by $A \mapsto \begin{cases} 1 & \text{if } A \in \mathcal{F} \\ 0 & \text{if } A \in \hat{\mathcal{F}} \end{cases}$ is a finitely additive 0/1-valued measure on $\mathcal{A}$. Furthermore,
if $\mathcal{F}$ is an ultrafilter, then $\mathcal{A} = \mathcal{P}(\mathbb{N})$, so $\mu_{\mathcal{F}}$ is an ultrafilter-measure. Also, $\mu_{\mathcal{F}}$ is a Dirac measure exactly when $\mathcal{F}$ is principal.

Example. The collection of cofinite subsets of $\mathbb{N}$ is a nonprincipal filter, but it is not ultra because it doesn't contain any infinite and co-infinite set. This filter is called the Fréchet filter.

Theorem. There is a nonprincipal ultrafilter on $\mathbb{N}$.

Proof. Start with the Fréchet filter $\mathcal{F}$, so it is already nonprincipal. By Zorn's lemma, there is a maximal filter $\mathcal{U} \supseteq \mathcal{F}$, if $\mathcal{U}' \supseteq \mathcal{U}$ is a filter, then $\mathcal{U}' = \mathcal{U}$.
We show that this $\mathcal{U}$ is ultra. Fix $A \subseteq \mathbb{N}$ and suppose $A \notin \mathcal{U}$. If $A \cap U = \emptyset$ for some $U \in \mathcal{U}$, then $A^c \supseteq U$ hence by (i), $A^c \in \mathcal{U}$. Thus suppose that $\mathcal{U} \cup \{A\}$ satisfies the finite intersection property, i.e. (ii). Then we let $\mathcal{U}'$ be the filter generated by $\mathcal{U} \cup \{A\}$, so $\mathcal{U}' \supsetneq \mathcal{U}$ and is a filter, contradicting the maximality of $\mathcal{U}$. $\square$

For a bdd sequence $x \in \ell^\infty$, x admits a convergent subsequence by Bolzano–Weierstrass. One can use a nonprincipal ultrafilter to "coherently choose" one such subsequence for every bdd sequence.

Def. Let μ be a finitely additive 0/1-valued measure on an algebra $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ (i.e. $\mathcal{F}_\mu$ is a filter). We say that the **limit** of x **along** μ is $L \in \mathbb{C}$, and write $\lim_{n \to \mu} x(n) = L$, if $\forall \varepsilon > 0$ $|x(n) - L| < \varepsilon$ for μ-a.e. $n \in \mathbb{N}$, i.e. $\mu(\{n \in \mathbb{N} : |x(n) - L| < \varepsilon\}) = 1$.

Examples. (a) The usual notion of limit $\lim_{n \to \infty} x(n)$ is the same as limit along the Fréchet filter.
(b) For a Dirac measure δ_{n_0}, $n_0 \in \mathbb{N}$, $\lim_{n \to \delta_{n_0}} x(n) = x(n_0)$.

Prop. Limits of bdd sequences along ultrafilters always exist. (HW.)

Having a non-Dirac ultrafilter measure (i.e. a nonprincipal ultrafilter) μ on $\mathcal{P}(\mathbb{N})$, we can define a mean on $\ell^\infty_{\mathbb{C}}$ as follows: for each $f \in \ell^\infty$,

$$\tilde{L}(x) := \lim_{n \to \mu} \frac{1}{n+1} \sum_{i \leq n} x(i).$$

It is not hard to check that indeed $\tilde{L}(x)$ is a mean on ℓ^∞, giving a different proof of the existence of means on ℓ^∞.